THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XV. JUNE-JULY, 1908. NOS. 6-7.

GENERALIZATION OF POSITIVE AND NEGATIVE NUMBERS.

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In the discussion of ordinary complex numbers it is generally made clear that the real numbers constitute a very special class of complex numbers, but the fact that this class is composed of all the complex numbers which are either positive or negative does not appear to have received sufficient emphasis. Calling a the amplitude or argument of a complex number we may group together all the numbers for which a has the same numerical value (that is, all the numbers situated on the same ray of the plane pencil whose vertex is the origin) and call them the a numbers. When $a \doteq 0$, we obtain the positive numbers, and when $a = \pi$, the negative ones. Hence the question arises whether we should not use the terms *zero-numbers* and π-*numbers* in place of positive numbers and negative numbers, respectively. Regardless of whether such a change of terms would be desirable it may be profitable to consider the positive and negative numbers from this standpoint, and this is the main object of the present note.

If a_0 represents any particular value of a it is clear that the sum of two a_0-numbers is necessarily an a_0-number; that is, all the numbers of the same ray have the group property* with respect to addition, but they clearly do not form a group with respect to this operation. The totality of numbers constituting the two rays a_0 and $a_0 + \pi$ evidently form a group with respect to either of the two operations, addition and subtraction, while the totality obtained by multiplying any one of them by all the real integers forms a subgroup with respect to either of these operations. With respect to the operations of addition and subtraction the two rays of real numbers do not present any group properties which differ from those of any other two rays which together form a straight line, but this is not the case with respect to the operations of multiplication and division as will appear more clearly from the following considerations.

It should be observed that the term division is here used with its most common meaning; viz., as the inverse of multiplication. Since a group in-

*Cf. Bocher, *Introduction to Higher Algebra*, 1907, p. 82.

volves the inverse of each of its operations it results that if a system of numbers forms a group when combined with respect to one of the fundamental operations of arithmetic it must also form a group with respect to the inverse of this operation. That is, if a system of numbers forms a group with respect to addition it also forms a group with respect to subtraction, and if it forms a group with respect to multiplication it also forms one with respect to division, and *vice versa.* The 0-numbers constitute the only ray of numbers which form a group with respect to multiplication. In fact, no other ray of numbers has even the group property with respect to either of the operations of multiplication and division, and if any finite number of rays have this property these rays must include the 0-numbers.

If any finite number of rays have the group property with respect to multiplication or division they evidently form a group with respect to multiplication but this is not necessarily true of special sets of numbers on such rays. For instance, the positive integers have the group property with respect to multiplication but they do not form a group with respect to this operation.* From the fact that the modulus of the product of two numbers is the product of the moduli of the factors it results that a system of numbers which form a group with respect to multiplication either includes only one number from a single ray or it includes an infinite number of numbers from every ray represented in the system. If the number of these numbers is finite they constitute the roots of an equation of the form $x^n=1$, and form a cyclic group since such an equation has primitive roots.

From the preceding paragraph it results that the rays as units form a cyclic group whenever numbers from a finite number of different rays are the elements of a group with respect to multiplication. *The ray of positive numbers is found in every such cyclic group,* while the ray of negative numbers occurs only in those of even order. Hence the positive numbers play a unique rôle in the groups of multiplication. While the ray of negative or π-numbers plays a less prominent rôle yet it enters into a larger number of the given finite groups of rays than any other ray except the one of positive numbers. A characteristic property of the positive and negative numbers is that they constitute the only two rays which form a group with respect to multiplication. It thus appears that while these two rays do not present any properties which differ from other straight lines of numbers with respect to the groups of addition they occupy a unique position in regard to the groups of multiplication; but their very special importance is due to the fact that they form a group with respect to both of the operations of addition and multiplication while no other finite number of rays has this property. This follows from the given results and the evident fact that *the pairs of rays which form a straight line constitute the only finite sets of rays of numbers which form a group with respect to addition.*

*Weber, *Lehrbuch der Algebra,* Vol. 2, 1899, p. 4. It is to be noted that the term ray is used to represent the simply infinite system of numbers represented by the points on such a ray.

The preceding considerations lead to some direct extensions of the most fundamental rules of operating with positive and negative numbers. For example, the rule for the sums of two real numbers with opposite signs is included in the following: To obtain the sum of a number on a given ray and a number on the extended ray* find the difference of their absolute values and prefix to this the angle of that one whose absolute value is the greater. The fact that the sign of the product of a positive and a negative number is negative is included in the statement that the angle of the product of a number on the α_0 ray and a number on its extension is $2\alpha_0+\pi$. From this it results that the necessary and sufficient condition that a number is real is that a negative number is obtained by multiplying it by a number on the extension of its ray, and the necessary and sufficient condition that a number is a pure imaginary is that a positive number is obtained when we multiply the number into any number on the extended ray.

In the light of these results the question whether the terms positive and negative, which do not exhibit any evidence of the fact that they represent two special values of the possible amplitudes of a number, should be replaced, at least in theoretic work, by terms which exhibit their places in the infinite series of amplitudes, assumes a deeper meaning. If one should be asked to defend the terms 0-numbers and π-numbers it would be merely necessary to reply that the adjectives express the values of the amplitudes of these numbers, but no such rational defense could be made for the terms positive numbers and negative numbers. At any rate this view point seems to deserve notice since the question involved is so fundamental, and the present note makes no other claims for usefulness or novelty than the presentation of a very elementary and important matter in a somewhat new light.

ON CERTAIN PROPERTIES OF THE ORBITS OF A PARTICLE SUBJECT TO A CENTRAL FORCE VARYING AS AN INTEGRAL POWER OF THE DISTANCE.†

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In this paper are discussed the orbits of a particle moving subject to a central force varying as an integral power of the distance. The discussion is made with regard to their concavity and convexity, the number and distances of their apses, the ranges of values through which their radial dis-

*If the angles of two rays differ by π each one of them is said to be the extension of the other.

†This discussion had its origin in an exercise given by Professor F. R. Moulton to his class in Celestial Mechanics. It was read before the Chicago Section of the American Mathematical Society, April 18, 1908.